ANALYTIC TRANSFORMATIONS OF EVERYWHERE DENSE POINT SETS*

BY

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I. Transformations of point sets

There is a well known theorem in the theory of point sets, due to Cantor,† to the effect that *All enumerable, everywhere dense linear point sets without first and last points have the same order type as the rational numbers.* That is, any set of this type can be mapped on the rational points of a line by a one to one correspondence which preserves order, and consequently any two sets of this type can be mapped on one another by such a correspondence.

A correspondence of two everywhere dense point sets clearly determines at most one continuous function which maps the segments on which the given sets are everywhere dense on one another, and also generates the correspondence. The requirement that the correspondence preserve order is equivalent to the requirement that a continuous mapping function exist, so that we may state the above theorem in the following form: *For any two enumerable linear point sets, each everywhere dense on an open interval, a continuous function can be found which maps the two intervals on one another, and effects a one to one correspondence between the point sets.*

Since the function of this theorem is by no means uniquely determined, the question naturally arises as to whether we can place further restrictions on it without destroying the validity of the theorem. It turns out that we may always require the function which effects the mapping to be *analytic* and it is the demonstration of this fact and some related questions which occupy our attention in this paper.

II. Existence of an analytic transformation

In proving the existence of an analytic mapping function, there is obviously no loss of generality in restricting the two given point sets to lie on the interval from 0 to 1. For a set on the interval a to b is mapped on this unit interval by the transformation $w = (x-a)/(b-a)$, one on the interval a to ∞ by the transformation $w = (x-a)/(1+x-a)$, and

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† Mathematische Annalen, vol. 46 (1895), p. 505.

one on the entire straight line by the transformation $w = e^x/(1+e^x)$. Consequently, if we show that two sets of the specified type on the unit interval may always be mapped on one another by an analytic transformation, the combination of one of the three transformations just given, the transformation for the two unit intervals, and the inverse of one of the three will yield an analytic transformation for any two intervals.

Consider then two point sets, each of which is enumerable and everywhere dense on the unit interval. Since the sets are enumerable, we may designate the points of the first set as

$$a_1, a_2, a_3, \cdots$$

and those of the second as

$$b_1, b_2, b_3, \cdots.$$

We shall make use of a set of small positive constants

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots,$$

selected so that their sum converges:

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots = h \qquad (h < 1)$$

but otherwise arbitrary.

Our method of setting up the mapping function will be one of successive approximation, each new approximation making our function behave properly at a new point, without affecting its behavior at points already considered. We start then with the function

$$y_1 = x,$$

which takes the end points of the two intervals into one another. This function takes the point a_1 into a_1. In general, this is not a b_i. Since, however, the b_i are everywhere dense on the unit interval, we may find a b_i as close to the point a_1 as we please, in particular, a b_{i_1} such that

$$b_{i_1} - a_1 = e_1 \frac{a_1(a_1 - 1)}{2}; \qquad |e_1| < \varepsilon_1.$$

From this we form our second approximation,

$$y_2 = x + e_1 \frac{x(x-1)}{2}.$$

It will be noted that this takes the unit interval into itself, in a one to one manner (since it has a positive derivative in this interval), and also takes a_1 into b_{i_1}.

If now we give y_2 the value b_1 (or b_2, if $b_{i_1} = b_1$), it will correspond to a single value of x in the unit interval, say x_1, which in general is not an a_i. But, in virtue of the fact that the a_i are everywhere dense, we may find an a_i distinct from a_1 as close to x_1 as we please, in particular, since y_2 is a continuous function, an a_{j_1} such that

$$y_2(a_{j_1}) - b_1 = k_1 \frac{b_1(b_1-1)(b_1-b_{i_1})}{3}; \qquad |k_1| < \varepsilon_1.$$

This enables us to form our third approximation,

$$y_3 + k_1 \frac{y_3(y_3-1)(y_3-b_{i_1})}{3} = x + e_1 \frac{x(x-1)}{2}.$$

As the derivative of the left member with respect to y_3 is positive in the unit interval, while that of the right member with respect to x is also positive in this interval, the function $y_3(x)$ maps the unit interval on itself in a one to one manner; it also obviously takes a_1 into b_{i_1} and a_{j_1} into b_1.

We next add a term to the right member to make a_2 (or a_3 if $a_{j_1} = a_2$) correspond to a b_i, say b_{i_2}:

$$e_2 \frac{x(x-1)(x-a_1)(x-a_{j_1})}{4}, \qquad |e_2| < \varepsilon_2.$$

Then we add a small term to the left member to make b_2 (or the b with smallest index not already used) correspond to an a_i, say a_{j_2}:

$$k_2 \frac{y(y-1)(y-b_{i_1})(y-b_1)(y-b_{i_2})}{5}, \qquad |k_2| < \varepsilon_2.$$

The method of procedure is now clear. At each stage we take the next a_i or b_i as the case may be, which has not been already used, and so change the corresponding member of the approximating equation that it shall correspond to a point of the other set for the new function. The changed term is in the form of a polynomial which vanishes at all the points already adjusted, and a numerical factor is inserted to make it, as well as its derivative, less in absolute value than the corresponding ε_n

throughout the interval 0 to 1. The process determines an equation each of whose sides is an infinite series:

$$y+\sum_{1}^{\infty} k_n \frac{y(y-1)(y-b_{i_1}) \cdots (y-b_{i_n})}{2n+1} = x+\sum_{1}^{\infty} e_n \frac{x(x-1)(x-a_1) \cdots (x-a_{j_n})}{2n+2}.$$

Let us consider these two series in turn. In the interval 0 to 1, each term in the right member is an analytic function of x, whose absolute value is less than the corresponding ϵ_n. Consequently, since the series of ϵ_n's converges, the right member represents an analytic function of x. Furthermore, since the series obtained by termwise differentiation also is dominated by the ϵ series, it represents the derivative of the function just obtained. As the sum of the ϵ series is h, less than unity, this derivative is always positive. Thus the right member is an increasing analytic function of x. Similarly the left member is an increasing analytic function of y. Thus the above equation determines y as an increasing analytic function of x, and accordingly maps the unit interval on itself in a one to one manner.

To find the transform of a point of the set a_i by this function, we note that since the a_i are enumerable, each a_i is reached at some stage of the approximating process. Thus all the terms after a certain one in the right member contain $x-a_i$ as a factor, and hence vanish when $x=a_i$. Also, from our method of procedure, there is a b_j which when substituted for y causes all the terms in the left member after a certain one to vanish, and makes the sum of those which do not vanish equal the right member with x replaced by a_i. Thus, since we already know that the transformation from x to y is one to one, b_j is the transform of a_i. Similar reasoning shows that each b_j has as its transform some a_i.

Having explicitly constructed a function with the desired properties, we may state

THEOREM I. *For any two enumerable linear point sets, each everywhere dense on an open interval, an analytic function can be found which maps the two intervals on one another, and effects a one to one correspondence between the point sets.*

We may remark in passing that if one of the intervals is infinite both ways, the function we have constructed is only analytic at the points of the open intervals in question; if, however, both intervals are semi-infinite or finite (not necessarily both of the same type) our function is analytic at the end points of the open interval as well (except, of course, for the pole in the semi-infinite case).

III. Approximation to an analytic function

If the intervals of the above theorem are both finite, we may put a further restriction on the analytic mapping function. In fact, we may have it approximate any given analytic function which maps the intervals on one another.

To see this, let us turn to the mapping function we have constructed in the preceding section which maps the unit interval into itself. We notice that it approximates the function with which we started,

$$y = x.$$

For, our final equation may be written in the form

$$F(y) + y = f(x) + x.$$

As both $F(y)$ and $f(x)$ are dominated by the ϵ series, and hence numerically less than h, we have

$$|y - x| = |f(x) - F(y)| \leqq |f(x)| + |F(y)| \leqq 2h.$$

Since h was entirely at our disposal, we can take it so that the final function approximates the original one to any desired degree.

A similar relation holds for the derivative, since from

$$F'(y)y' + y' = f'(x) + 1$$

we have

$$|y' - 1| = \left|\frac{f'(x) - F'(y)}{1 + F'(y)}\right| \leqq \frac{2h}{1-h},$$

as $f'(x)$ and $F'(y)$ are both numerically less than h from their definition. Thus y' can be made to approximate unity, by a proper choice of h.

Suppose, now, we were given two enumerable point sets, each everywhere dense on some finite, open interval and an analytic function, $g(x)$, which mapped one of the intervals on the other, and had a derivative which was positive in the corresponding closed interval. We could start with the function

$$y_1 = g(x)$$

and build up a series of approximating functions as in Section II which would map one of the point sets on the other. Of course, h would have to be taken less than the minimum value of $g'(x)$ in the interval, to make the approximations monotonic. If we also took $h < \eta/2$, we would find that, for the final function,

$$|y - g(x)| < \eta.$$

This establishes

THEOREM II. *For any two enumerable linear point sets, each everywhere dense on an open interval, and any analytic function which maps one of the corresponding closed intervals on the other, its derivative being positive in this closed interval, an analytic function can be found which maps the two intervals on one another, effects a one to one correspondence between the point sets, and approximates the given function uniformly.*

For the function we have just constructed, we would also have

$$|y' - g'(x)| \leqq \frac{h(1+G)}{1-h},$$

where G is an upper bound for $g'(x)$. Thus h can be chosen so as to make the derivative of the new function approximate $g'(x)$. As we constructed our function, only the first derivatives of the series appearing in the final equation are dominated by the ε series, and hence less than h. By replacing the numerical factors in the denominators of the separate terms by factorials, we can arrange that all such derivatives are so dominated. This enables us to write down equations for the higher derivatives of somewhat similar form to that just given for the first one. Then h can be chosen so as to make any given number of derivatives approximate those of the given function (not an indefinite number, since $(1-h)^m$ appears in the denominator of the bound for the mth derivative), which leads to the

COROLLARY. *The function of Theorem* II *may be so chosen that its first m derivatives* (*m being any number*) *approximate those of the prescribed analytic function uniformly.*

IV. APPROXIMATION TO A CONTINUOUS FUNCTION

Instead of starting out with an analytic function which maps our two intervals on one another, we may start with one which is merely continuous, and seek an analytic function which approximates this and takes our two sets into one another. As we have already shown how to approximate to an analytic function of a certain type, we need merely approximate to the continuous function by one of this type. As the analytic function must map the same interval as the continuous function, i. e., have the same initial and final values, and have a positive derivative throughout the interval, we may not apply the Weierstrass theorem directly, but need an extension of it, to which we proceed.

LEMMA. *Any continuous function which maps one interval on another in a one to one manner preserving sense may be approximated uniformly by an analytic function with positive derivative which maps these intervals on one another.*

The initial step in constructing the function required is to approximate the given continuous function, $c(x)$, by a broken line function, $B(x)$. We take it with the same end points, so that

$$B(a) = c(a), \qquad B(b) = c(b),$$

where a and b are the end points of the interval considered; and also so that throughout the interval

$$|B(x) - c(x)| < \eta/4$$

where η is to be a measure of the final approximation. Since $c(x)$ mapped the intervals on one another in a one to one manner, the segments forming $B(x)$ may be so taken (e. g. as the chords of an inscribed polygon) that their slopes are all positive.

We may obtain an approximation with a continuous derivative by replacing the ends of the chords by small circular arcs, tangent to the chords. They may be taken so small that, if $E(x)$ is the new function,

$$E(x) - B(x)| < \eta/4.$$

$E(x)$ has a continuous derivative in the closed interval which is always positive. It therefore has a positive minimum, s, so that

$$E'(x) > s > 0.$$

Since the function $E'(x)$ is continuous, by the theorem of Weierstrass, it can be approximated uniformly by an analytic function. Let, then, $F(x)$ be an analytic function, such that

$$|F(x)-E'(x)|<\zeta,$$

where

$$\zeta<s/3 \qquad \text{and} \qquad \zeta<\frac{\eta}{4(b-a)}.$$

Finally, we put

$$G(x) = c(a)+\int_a^x F(x)\,dx+\frac{(x-a)}{b-a}\left[c(b)-c(a)-\int_a^b F(x)\,dx\right].$$

The function $G(x)$ is clearly analytic, and from its form agrees with $c(x)$ at the points a and b. Furthermore,

$$G'(x) = F(x)+\frac{1}{b-a}\left[c(b)-c(a)-\int_a^b F(x)\,dx\right].$$

Since

$$|F(x)-E'(x)|<s/3 \qquad \text{and} \qquad E'(x)>s,$$

$$F(x)>2s/3.$$

Also

$$\left|c(b)-c(a)-\int_a^b F(x)\,dx\right| = \left|\int_a^b [E'(x)-F(x)]\,dx\right|<\frac{s(b-a)}{3}.$$

Consequently

$$G'(x)>2s/3-s/3 = s/3>0,$$

so that $G(x)$ has a positive derivative.

Finally, since

$$E(x) = c(a)+\int_a^x E'(x)\,dx, \qquad \text{and} \qquad c(b)-c(a) = \int_a^b E'(x),$$

$$G(x)-E(x) = \int_a^x [F(x)-E'(x)]\,dx + \frac{x-a}{b-a}\int_a^b [E'(x)-F(x)]\,dx$$

and we have

$$|G(x)-E(x)|<(x-a)\zeta+\frac{x-a}{b-a}(b-a)\zeta<\eta/2.$$

This, combined with our earlier inequalities for $E(x)$ and $B(x)$, shows that

$$|G(x)-c(x)|<\eta,$$

and accordingly $G(x)$ may be taken as the function demanded by the lemma.

By combining the lemma with Theorem II, that is, using the lemma to approximate a continuous function by an analytic function, and then using this as the given analytic function of Theorem II, we obtain

THEOREM III. *For any two enumerable linear point sets, each everywhere dense on an open interval, and any continuous function which maps one of the corresponding closed intervals on the other in a one to one manner which preserves sense, an analytic function can be found which maps the two intervals on one another, effects a one to one correspondence between the point sets, and approximates the given function uniformly.*

One case of this theorem deserves to be specially mentioned. That is, the case in which the two enumerable sets of points become all the rational points in the intervals in question. While we have previously kept the initial and final values of the given function unchanged, it is evident that we can always change the given function by an amount as small as we wish, and bring it about that the initial and final values of the function are rational or irrational according as those of the argument are. This enables us to state

THEOREM IV. *Any continuous function, monotonic in an interval (actually increasing or decreasing, not stationary) may be approximated uniformly in this interval by an analytic function which takes on rational values when, and only when, its argument is rational.*

V. EXTENSIONS TO NON-LINEAR POINT SETS

The theorems we have stated thus far relate to sets of points on segments of straight lines. Similar theorems may be formulated for sets of points on analytic arcs, since by the definition of such an arc it may be mapped on a straight line by an analytic function, and this mapping clearly takes an enumerable everywhere dense set of points on the arc into another such set on the straight line.

If we attempt to extend the theorems to sets of points everywhere dense in a two-dimensional region, we meet difficulties. For, in the process of Section II as applied to an interval, we kept the end points fixed, and thus insured at each stage that the transforms of new points by the function then reached were actually in the region where the points were everywhere dense. As we can not hold the boundary points fixed for a two-dimensional region, the process is no longer applicable. That the proposed generalization of the theorem itself, as well as the method of proof, breaks down, can be seen from a very simple example. Consider two sets of points, each enumerable and everywhere dense inside the unit circle. Any transformation which took one of these sets into the other in a one to one and continuous manner would necessarily be continuous on the boundary of the circle when extended to all the points in and on the circle. If, now, it was analytic, it would necessarily be a linear fractional transformation, as easily follows from the known theorems on conformal mapping. Let the first set be composed of all the rational points in the unit circle, and the second set consist of all the rational points in the circle and one irrational point. There is no analytic transformation which will take the first set into the second. For, by what has been said, it would have to be a linear fractional transformation. Hence it would preserve the value of the anharmonic ratio of four points. But this ratio is rational for all the points of the first set, while for some groups of points in the second, containing the irrational point, it would be irrational.

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